

Simulaciones numéricas de un modelo macroscópico de tráfico vehicular con relajación

Numerical simulations of a macroscopic traffic Flow model with relaxation

DOI: <https://dx.doi.org/10.17981/ingecuc.20.2.2024.18>

Artículo de Investigación Científica.

Fecha de Recepción: 02/10/2023, Fecha de Aceptación: 15/12/2023.

Carlos Arturo Vega-Fuentes 

Universidad del Norte. Barranquilla, (Colombia)
cvega@uninorte.edu.co

To cite this paper

C. Vega-Fuentes, "Numerical simulations of a macroscopic traffic Flow model with relaxation," *INGE CUC*, vol. 20, no. 2, 2024. DOI: <https://dx.doi.org/10.17981/ingecuc.20.2.2024.18>

Resumen

Introducción: Las simulaciones para modelos de tráfico vehicular constituyen una activa línea de investigación en la actualidad debido al creciente flujo de tráfico en las urbes.

Este trabajo se enfoca en aproximar numéricamente las soluciones de un modelo macroscópico descrito por una ley de balance. Teniendo presente que, en general, no es posible hallar soluciones analíticas del problema, se hace necesario hallar aproximaciones numéricas confiables y eficientes que eventualmente puedan ser usadas en escenarios del mundo real.

Objetivo: El objetivo de este trabajo es aproximar numéricamente las soluciones de un modelo de tráfico vehicular con término de relajación utilizando métodos de alto orden tanto para la discretización espacial como para la temporal.

Metodología: Se realiza primero la discretización de la derivada espacial usando un método esencialmente no oscilatorio ponderado de quinto orden (para soluciones regulares) obteniéndose un esquema semi-discreto. Luego, se discretiza la derivada temporal usando un método de Runge-Kutta exponencial desarrollado recientemente que preserva el signo y las soluciones de estado estacionario.

Resultados: Se obtienen aproximaciones de las soluciones para tres experimentos numéricamente exigentes que corresponden a problemas de Riemann y se compara la solución aproximada obtenida en uno de ellos con la solución analítica conocida para un caso particular, observándose una excelente concordancia entre las dos soluciones y un comportamiento consistente de los errores.

Conclusiones: Se usó un método de quinto orden espacial combinado con un método de discretización tipo exponencial de tercer orden para calcular las soluciones numéricas de un conocido modelo de tráfico vehicular descrito por un sistema de dos ecuaciones diferenciales parciales con término de relajación. El esquema es completamente explícito y se verificó que captura correctamente los distintos tipos de ondas que se generan: choques, discontinuidades de contacto, rarefacciones y rarefacciones generalizadas.

Palabras claves

Modelos de tráfico vehicular; leyes de balance; solución numérica; ondas de rarefacción; ondas de choque.

Abstract

Introduction: Simulations for traffic flow models are an active line of research nowadays due to the increasing traffic flow in cities. This work focuses on numerical approximations of the solutions of a macroscopic model described by a balance law. In general, it is not possible to find analytical solutions to the problem considered, then it is necessary to construct reliable and efficient numerical approximations that can eventually be used in real-world scenarios.

Objective: The aim of this paper is to approximate numerically the solutions of a traffic flow model with relaxation term using high-order methods for both the spatial discretization and the temporal discretization.

Method: First, the spatial derivative is discretized by using a fifth-order (for smooth solutions) weighted essentially non-oscillatory method to obtain a semi-discrete scheme. Then, the temporal derivative is discretized using a recently developed exponential Runge-Kutta method which preserves the sign and steady-states solutions.

Results: Approximate solutions are obtained for three numerically demanding experiments corresponding to Riemannian problems and the approximate solution obtained in one of them is compared with a given analytical solution for a particular case, observing an excellent agreement between the two solutions and consistent error behavior.

Conclusions: A spatial fifth-order method combined with an exponential third-order exponential-type discretization method was used to calculate the numerical solutions of a well-known vehicular traffic model described by a system of two partial differential equations with relaxation term. The scheme is fully explicit, and it was verified that it correctly captures the different types of waves generated: shocks, contact discontinuities, rarefactions and generalized rarefactions.

Keywords

Traffic Flow models; Balance Laws; numerical solutions; rarefactions waves; shock waves.



INTRODUCCIÓN

Currently, the study of traffic flow models constitutes a very active line of research due to the growing number of vehicles in urban areas and the imperative need to improve traffic conditions. From a mathematical point of view, models to describe and simulate vehicular traffic can be classified into two large groups [1]: microscopic models and macroscopic models. The former describe the behavior (movement) of each vehicle based on the behavior of the vehicle or vehicles in front of it. Although microscopic models provide a very realistic description of traffic flow, they usually require considerable computational effort to obtain numerical results when there is a high volume of traffic because such models involve the individual behavior of each vehicle simultaneously. On the other hand, macroscopic models do not study the individual behavior of each vehicle but rather describe traffic as continuous flow. Macroscopic models were first introduced by Lighthill and Whitham [2] and by Richards [3]. This first model, called the LWR model, is described by the following first order differential equation obtained from the continuity equation for density of vehicles:

$$\partial_t \rho + \partial_x (\rho v) = 0 \quad (1)$$

where $\rho(x,t)$ and $v(x,t)$ denote, respectively, the local density of vehicles (veh/m) and is the average speed (m/s) at position and at time . The LWR model is useful for characterizing equilibrium traffic flows in which speed is a function of traffic density, i.e., all drivers adjust their speed depending on the spacing between vehicles. However, the LWR model fails to describe other traffic phenomena such as accelerations and decelerations or non-equilibrium conditions, for example, instantaneous speed changes. To overcome these difficulties, the so-called second-order models have been proposed, which are described by a system of two partial differential equations, one of which includes a relaxation term describing drivers' speed adjustments due to traffic conditions ahead of them. One of the most famous second-order macroscopic models is the Aw-Rascle model [4]-[5]. This model is described by the scalar equation (1) together with an equation describing the mass flow variations due to the road conditions in front of the driver. More precisely, the model in conservative form is described by the following system of partial differential equations

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t (\rho w) + \partial_x (\rho v w) = \frac{1}{T} g(\rho, w) \end{cases} \quad (2)$$

where $w=v+P(\rho)$ with $P(\rho)$ a given increasing function describing the anticipation of road conditions in front of the drivers, $g(\rho,w)$ is a relaxation which considers possible entrances or exits on the road and the parameter T is a relaxation time.

By introducing the variable $z=\rho w$ (called generalized momentum), the system (2) can be written as the one-dimensional balance law

$$\partial_t U + \partial_x F(U) = S(U) \quad (3)$$

where $U=(\rho,z)$ is the vector of conservative variables, $F(U)=(\rho v, \rho v w)=(z-\rho P(\rho), z^2/\rho - zP(\rho))$ is the flux function and $S(U)=(0, g(\rho,w)/T)$ is the source term. Characteristic speeds are the eigenvalues of the Jacobian matrix of the flux function and are given by

$$\begin{aligned} \lambda_1 &= \frac{z}{\rho} - P(\rho) - \rho P'(\rho) \\ \lambda_2 &= \frac{z}{\rho} - P(\rho) \end{aligned} \quad (4)$$

Since $P(\rho)$ is an increasing function, then the characteristic speeds are distinct and, consequently, the system (3) is hyperbolic.

Since, in general, it is not possible to explicitly find an analytical solution of the system (2), the alternative is to find numerical approximations. However, the development of efficient

numerical methods is a complex task due to the presence of the relaxation parameter in the source term given by (2) imposes some degree of stiffness on the system. This implies that when performing the time discretization using standard explicit methods, the time step is very restrictive which affects the efficiency of the method. A very popular alternative that avoids this restriction is the implicit-explicit Runge-Kutta (IMEX-RK) schemes, in which the hyperbolic part is treated explicitly and the source term implicitly [6]. Due to the nonlinearity of the source term in (2), if such a term is treated implicitly, then it is necessary to solve nonlinear algebraic equations at each time step, thus increasing the computational effort. On the other hand, systems of type (3) usually admit steady state solutions in which the balance between the gradient of the flux function and the source term must be maintained. However, most IMEX schemes do not satisfy this requirement [7]. A very attractive technique for temporal discretization to overcome the disadvantages of the aforementioned schemes are the high-order Runge-Kutta exponential methods introduced in [8] and recently improved in [9]. A detailed description of these methods and the schemes used for spatial discretization will be given in the next section.

METHODOLOGY

Since in general it is not possible to find an exact or analytical solution of the system (3), it is imperative to approximate the solution numerically. For this purpose, a discretization of the spatial derivative of the flux function will first be performed using the robust fifth-order weighted essentially non-oscillatory (WENO) method [10]-[11]. This method is especially useful for approximating hyperbolic partial differential equations whose solutions may be discontinuous as is the case of the model considered in this work.

We start by discretizing the computational domain $[a, b]$ where a and b are the ends of the portion of the road considered for the study. For simplicity, we take a uniform mesh, i.e., we perform a regular partition of $[a, b]$ into M cells $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ with mesh width $h = (b - a)/M$ and centers at the points $x_j = a + jh, j = 1, 2, \dots, M$. Thus, a semi-discrete scheme for approximating the solution of the system (3) has the form

$$\frac{dU_j(t)}{dt} = -\frac{1}{h} \left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} \right) + S_j \quad (5)$$

Here, $U_j(t)$ denotes the numerical approximation of $U(x_j, t)$, $\hat{f}_{j+\frac{1}{2}}$ is a numerical flux at the interface $x_{j+\frac{1}{2}}$ and $S_j = (0, s_j)$ is the discretization of the source term, where

$$s_j = g(\rho_j, z_j)/T.$$

Note that the first term on the right-hand side in (5) corresponds to the discretization of the spatial derivative $\partial_x f(U)$ and such a vector will be denoted by $(L_j(U), M_j(U))$. To compute $\hat{f}_{j+\frac{1}{2}}$, the widely used fifth order WENO method is used. For the sake of clarity in presentation we next describe in detail the construction of $\hat{f}_{j+\frac{1}{2}}$. Due to nonlinearity of the model, it is necessary to split the flux into positive and negative parts, that is, $\mathbf{F}(U) = \mathbf{F}^+(U) + \mathbf{F}^-(U)$. In this paper we use the Lax-Friedrichs flux splitting $F^\pm(U) := \frac{1}{2}(F(U) \pm \gamma U)$. Here, γ is the so-called viscosity coefficient or maximal characteristic velocity which is calculated from the eigenvalues of the Jacobian matrix given in (4) by using the expression

$$\Gamma = \max_j \left\{ \left| \lambda_1 \left(U_{j+\frac{1}{2}} \right) \right|, \left| \lambda_2 \left(U_{j+\frac{1}{2}} \right) \right| \right\} \quad (6)$$

where $U_{j+\frac{1}{2}} = \frac{1}{2}(U_j + U_{j+1})$.

Next, the numerical fluxes $\hat{f}_{j+\frac{1}{2}}^+$ and $\hat{f}_{j+\frac{1}{2}}^-$ are constructed by using the smoothness indicators IS_p^+ y IS_p^- for $p = 0, 1, 2$, which are given by the following equations

$$\begin{aligned}
\text{IS}_0^+ &= \frac{13}{12} \left(f_{j-2}^+ - 2f_{j-1}^+ + f_j^+ \right)^2 + \frac{1}{4} \left(f_{j-2}^+ - 4f_{j-1}^+ + 3f_j^+ \right)^2 \\
\text{IS}_1^+ &= \frac{13}{12} \left(f_{j-1}^+ - 2f_j^+ + f_{j+1}^+ \right)^2 + \frac{1}{4} \left(f_{j-1}^+ - f_{j+1}^+ \right)^2 \\
\text{IS}_2^+ &= \frac{13}{12} \left(f_j^+ - 2f_{j+1}^+ + f_{j+2}^+ \right)^2 + \frac{1}{4} \left(3f_j^+ - 4f_{j+1}^+ + f_{j+2}^+ \right)^2
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\text{IS}_0^- &= \frac{13}{12} \left(f_{j+1}^- - 2f_{j+2}^- + f_{j+3}^- \right)^2 + \frac{1}{4} \left(3f_{j+1}^- - 4f_{j+2}^- + f_{j+3}^- \right)^2 \\
\text{IS}_1^- &= \frac{13}{12} \left(f_j^- - 2f_{j+1}^- + f_{j+2}^- \right)^2 + \frac{1}{4} \left(f_j^- - f_{j+2}^- \right)^2 \\
\text{IS}_2^- &= \frac{13}{12} \left(f_{j-1}^- - 2f_j^- + f_{j+1}^- \right)^2 + \frac{1}{4} \left(f_{j-1}^- - 4f_j^- + 3f_{j+1}^- \right)^2
\end{aligned} \tag{8}$$

Next, the smoothness indicators are used to compute the weights σ_p^+ y σ_p^- for $p = 0, 1, 2$:

$$\sigma_p^+ = \frac{\theta_p^+}{\theta_0^+ + \theta_1^+ + \theta_2^+}; \sigma_p^- = \frac{\theta_p^-}{\theta_0^- + \theta_1^- + \theta_2^-} \tag{9}$$

where

$$\theta_p^+ = \frac{C_p}{(\delta + \text{IS}_p^+)^2}; \theta_p^- = \frac{C_p}{(\delta + \text{IS}_p^-)^2} \tag{10}$$

The coefficients C_p for $p = 0, 1, 2$, are the ideal weights [12] which are given by $C_0 = \frac{1}{10}$, $C_1 = \frac{6}{10}$, and $C_2 = \frac{3}{10}$. The positive parameter δ was originally introduced to avoid null denominator. However, as it is pointed out in [12], large values of may cause oscillations, while small values may cause an order drop. In this paper, this parameter is taken as $\delta=10^{-40}$ which corresponds to the recommended value in [12].

Finally, the numerical flow is formed by adding the positive and negative parts, that is,

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}}^+ + \hat{f}_{j+\frac{1}{2}}^- \tag{11}$$

So far only the spatial discretization of (3) has been considered, leaving the time variable continuous. For the discretization of the time derivative in scheme (5) a type of exponential Runge-Kutta method (abbreviated RK-ERK resulting from a combination of a standard third-order Runge-Kutta method and a modified exponential Runge-Kutta method) of third-order accuracy recently proposed in [9] will be used. The description of the method is summarized as follows. The computations of an RK-ERK scheme necessary to advance from $U^n = (\rho^n, z^n)$ which corresponds to the approximate value of the solution at time $t^n = n\Delta t$ to $U^{n+1} = (\rho^{n+1}, z^{n+1})$ which corresponds to the approximate value of the solution at time are given by

$$\begin{aligned}
\rho^1 &= \alpha_{10}\rho^n + \beta_{10}\Delta t L(\rho^n, z^n) \\
z^1 &= z^n + \frac{\beta_{10}\Delta t [M(\rho^n, z^n) + s(\rho^n, z^n)]}{A_1} \\
\rho^2 &= \alpha_{20}\rho^n + \alpha_{21}\rho^1 + \beta_{21}\Delta t L(\rho^1, z^1) \\
z^2 &= \frac{\alpha_{20}z^n}{A_2} + \frac{e^{\beta_{10}\mu\Delta t} \{ \alpha_{21}z^1 + \beta_{21}\Delta t [M(\rho^1, z^1) + s(\rho^1, z^1) + \mu z^1] \}}{A_2} \\
\rho^{n+1} &= \alpha_{30}\rho^n + \alpha_{31}\rho^1 + \alpha_{32}\rho^2 + \beta_{32}\Delta t L(\rho^2, z^2) \\
z^{n+1} &= \frac{\alpha_{30}z^n}{A_3} + \frac{e^{\beta_{10}\mu\Delta t} \alpha_{31}z^1}{A_3} + \frac{e^{A\mu\Delta t} (\alpha_{32}z^2 + \beta_{32}\Delta t [M(\rho^2, z^2) + s(\rho^2, z^2) + \mu z^2])}{A_3}
\end{aligned}$$

The optimal values of the coefficients are given in Table 1 (more details can be found in [9]).

$\alpha_{10}=0$	$\alpha_{21}=0.3313107066925596$	$\beta_{10}=0.7071933376925014$
$\alpha_{20}=0.6686892933074404$	$\alpha_{31}=0.2039576138780898$	$\beta_{21}=0.4178047564915065$
$\alpha_{30}=0.3487419430256090$	$\alpha_{32}=0.4473004430963011$	$\beta_{32}=0.5640754637100439$

TABLE 1. OPTIMAL COEFFICIENTS FOR THE RK-ERK METHOD.

In addition,

$$A = \alpha_{21}\beta_{10} + \beta_{21}, A_1 = \alpha_{10} + \beta_{10}\mu\Delta t, A_2 = \alpha_{20} + e^{\beta_{10}\mu\Delta t}(\alpha_{20} + \beta_{10}\mu\Delta t)$$

$$A_3 = \alpha_{30} + \alpha_{31}e^{\beta_{10}\mu\Delta t} + e^{A\mu\Delta t}(\alpha_{32} + \beta_{32}\mu\Delta t)$$

and the parameter μ is calculated dynamically based on Theorem 2.3 proved in [8], namely

$$\mu = \max\left\{\frac{-s(\rho^n, z^n)}{z^n}, 0\right\} \quad (12)$$

This choice of μ guarantees that the scheme satisfies the sign property, i.e., if $U^n > 0$ then $U^{n+1} > 0$. Such a property is especially important because it avoids unphysical negative densities in the approximate solution. Another remarkable advantage of the RK-ERK method is that it is not necessary to take a restrictive time step [9], unlike other explicit methods for which the presence of the stiffness source term forces to take a very small time step to guarantee stability. Furthermore, the RK-ERK method preserves the steady state solutions, i.e., if $(L_j(U^n), M_j(U^n)) + (0, s_j(U^n)) = 0$ then $U^{n+1} = U^n$.

NUMERICAL RESULTS

In this section, three examples are presented to obtain numerical solutions of the model (2) with Riemann-type initial conditions, i.e., piecewise constant initial data. In all examples a spatial mesh with $M = 200$ is taken. The units of length are given in meters (m) and time in seconds (s).

Example 1.

In this example, taken from [5] the relaxation function is $g(\rho, z) = \rho(V(\rho) - v) = \rho(V(\rho) + P(\rho)) - z$, where $V(\rho)$ is a function describing the dependence of the velocity with respect to the density for an equilibrium situation. This equilibrium velocity is decreasing with respect to density and is taken as the following fitting function for experimental data

$$V(\rho) = v_m \cdot \frac{\pi}{2 + \arctan\left(\frac{11(\rho - 0.22)}{\rho - 1}\right)} / \frac{\pi}{2 + \arctan(11 \cdot 0.22)} \quad (13)$$

where v_m is the normalized maximum speed. The initial conditions are

$$U(x, 0) = \begin{cases} U_l = (\rho_l, z_l) & \text{si } x \leq x_d \\ U_r = (\rho_r, z_r) & \text{si } x > x_d \end{cases} \quad (14)$$

where $x_d = 0$ is the point where the discontinuity is located with left and right states given by $U_l = (0.05, -0.2971)$ and $U_r = (0.05, -0.2746)$ respectively. As in [5], we also take $P(\rho) = 2 \ln(\rho)$ and the computational domain $[-250, 150]$. In Fig 1 the evolution of the density and momentum of the traffic flow ρv is shown (first column) along with the profiles of the solutions at $t = 1$ s (second column) with relaxation time $T = 0.2$ s. These results are contrasted with the numerical solutions for the same problem but without relaxation term ($T = \infty$) presented in Fig 2. In the latter case, the solutions at $t = 1$ s for both density and momentum of the traffic flow consist of a rarefaction wave followed by a contact discontinuity.

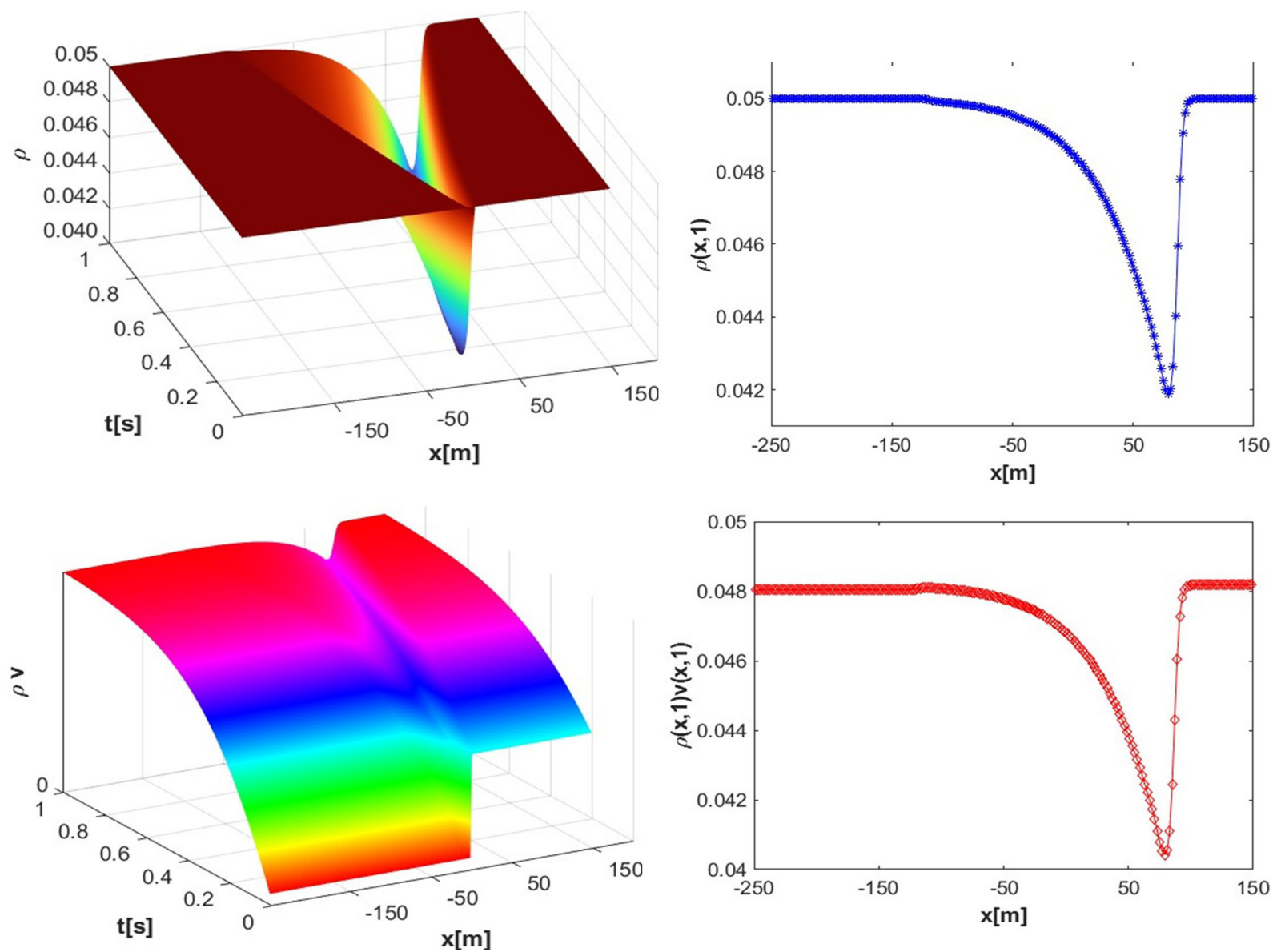


Fig. 1. Solutions evolution for density and momentum of the traffic Flow (left) with the corresponding solutions at $t = 1$ s (right) for example 1 with relaxation.
Source: author.

Example 2.

In this second scenario, the same relaxation function of example 1 is considered with $P(\rho) = \rho^2$, computational domain $[0, 20]$, relaxation time $T = 0.3$ s and initial conditions

$$U(x, 0) = \begin{cases} (0.5, 0.15) & \text{si } x \leq 10 \\ (0.15, 0.95) & \text{si } x > 10 \end{cases} \quad (15)$$

describing the situation of faster vehicles downstream followed by slower ones from behind. The numerical solutions corresponding to the time evolution are shown in Fig 3 along with the density and velocity profiles at $t = 8$ s.

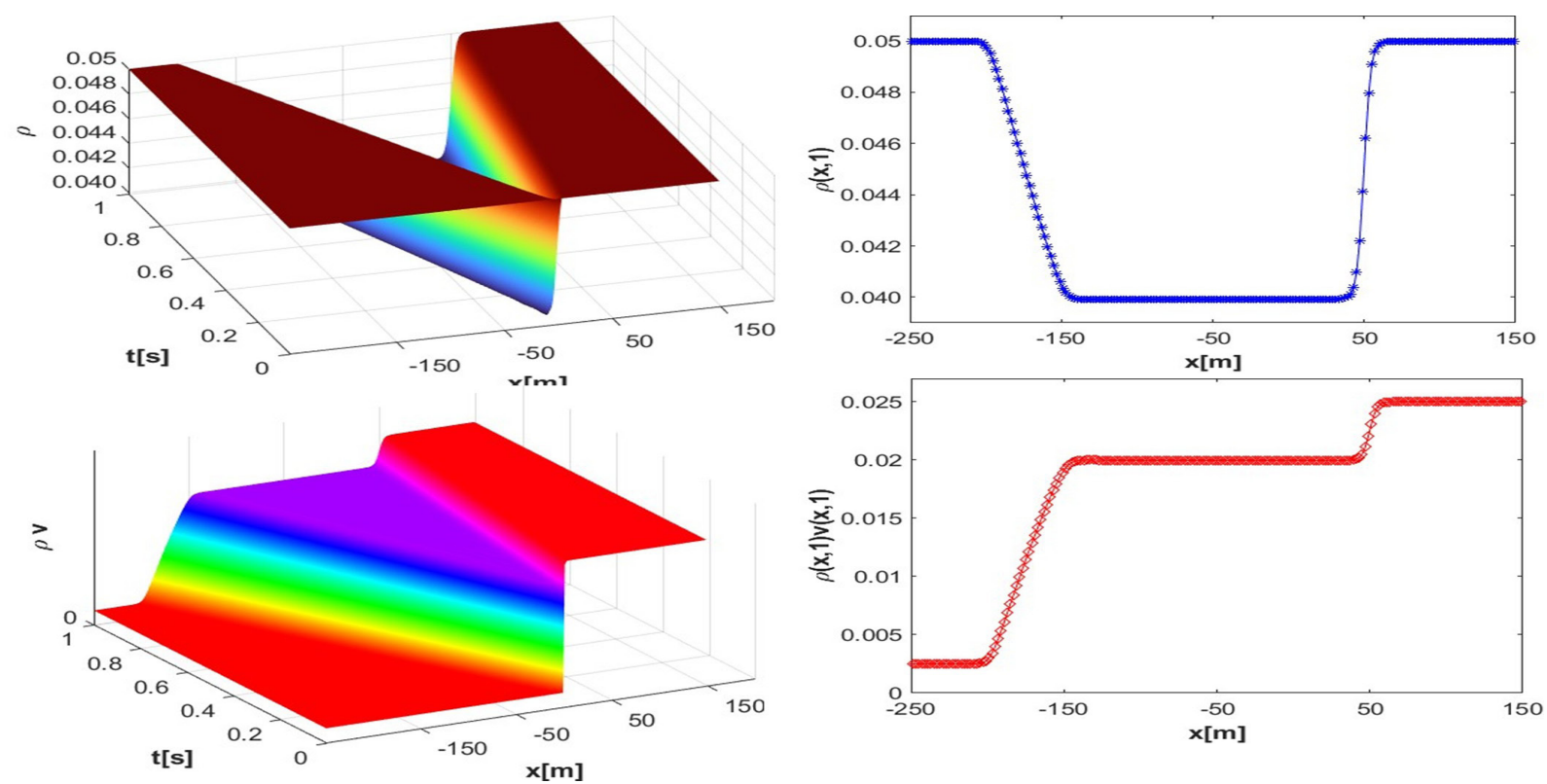


Fig. 2. Solutions evolution for density and momentum of the traffic Flow (left) with the corresponding solutions at $t = 1$ s (right) for example 1 without relaxation.
Source: author.

Example 3.

For this test case, we take $P(\rho)=\rho$, the relaxation time is $T=1$, the computational domain is the interval and the relaxation function is taken as $g(\rho, z) = (0.1(z - \rho P(\rho)) + 2\rho)\left(\frac{z}{\rho} - 9\right)$. The aim of this numerical experiment, whose parameters are taken from [13], is to use the numerical scheme described in this paper to simulate the double Riemann problem with initial data

$$U\left(x, 0\right)=\left\{\begin{array}{ll} U_1=\left(\rho_1, z_1\right) & \text { si } x < 0 \\ U_2=\left(\rho_2, z_2\right) & \text { si } 0 < x < L \\ U_3=\left(\rho_3, z_3\right) & \text { si } x > L \end{array}\right. \quad (16)$$

where $\rho_1 = \frac{13}{4}, \rho_2 = 7, \rho_3 = 4, z_1 = \frac{299}{8}, z_2 = 63$ and $z_3 = 36$. This situation corresponds to the traffic light problem where in there is a red traffic light which a switches to green. Double Riemann problems are demanding tests for testing numerical methods because their solutions may involve shock waves, contact discontinuities and generalized rarefaction waves. In [13] the exact solution of this problem expressed in terms of the density and velocity was obtained, namely

$$(\rho(x, t), v(x, t))=\left\{\begin{array}{ll} \left(\rho_1, v_1\right) & x < x_s(t) \\ \left(\hat{\rho}(x, t), \hat{v}(x, t)\right) & x_s(t) < x \leq x_r(t) \\ \left(\rho_3, v_3\right) & x \geq x_r(t) \end{array}\right. \quad (17)$$

Here, $\hat{x}_s(t) = (v_1 - \rho_1)t - 2L\frac{\sqrt{t}}{\sqrt{t_c}} + L, \hat{x}_r(t) = (v_3 - \rho_3)t + L, \hat{\rho}(x, t) = 0.5(v_1 + \rho_1 - \frac{x-L}{t}), \hat{v}(x, t) = v_2 + \rho_2 - \hat{\rho}(x, t)$. The solution given by (17) is valid for $t > t_c$ where $t_c = \frac{L}{v_1 - v_2}$ is the critical time and compared with the approximate solution in Fig 4

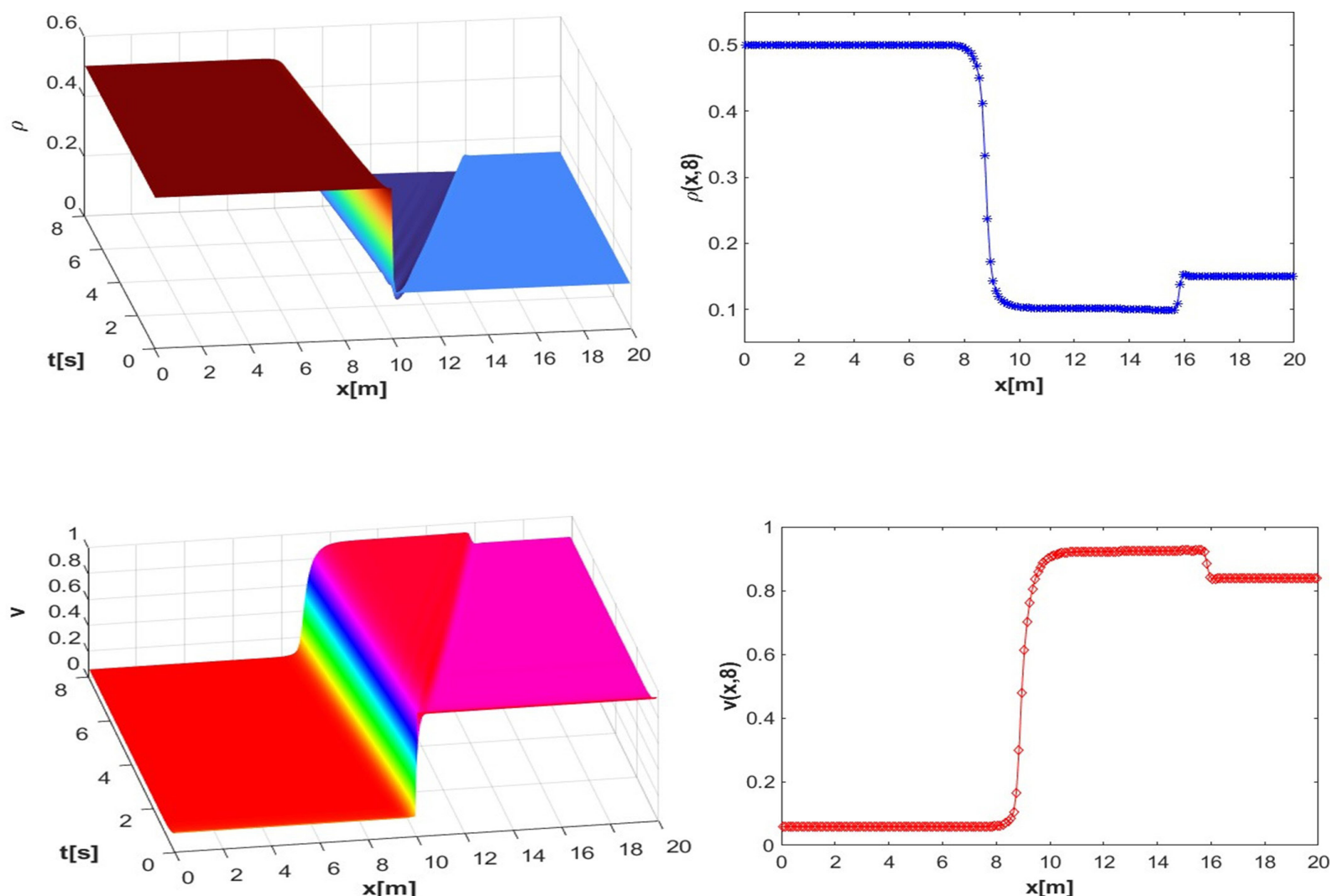


Fig. 3. Solutions evolution for density and velocity (left) with the corresponding solutions at $t = 8$ s for example Ej. 2. Source: autor.

In Fig 4 it can be observed that the numerical solution agrees very well with the analytical solution which corresponds to an initial shock and a generalized rarefaction wave propagating backward until they meet.

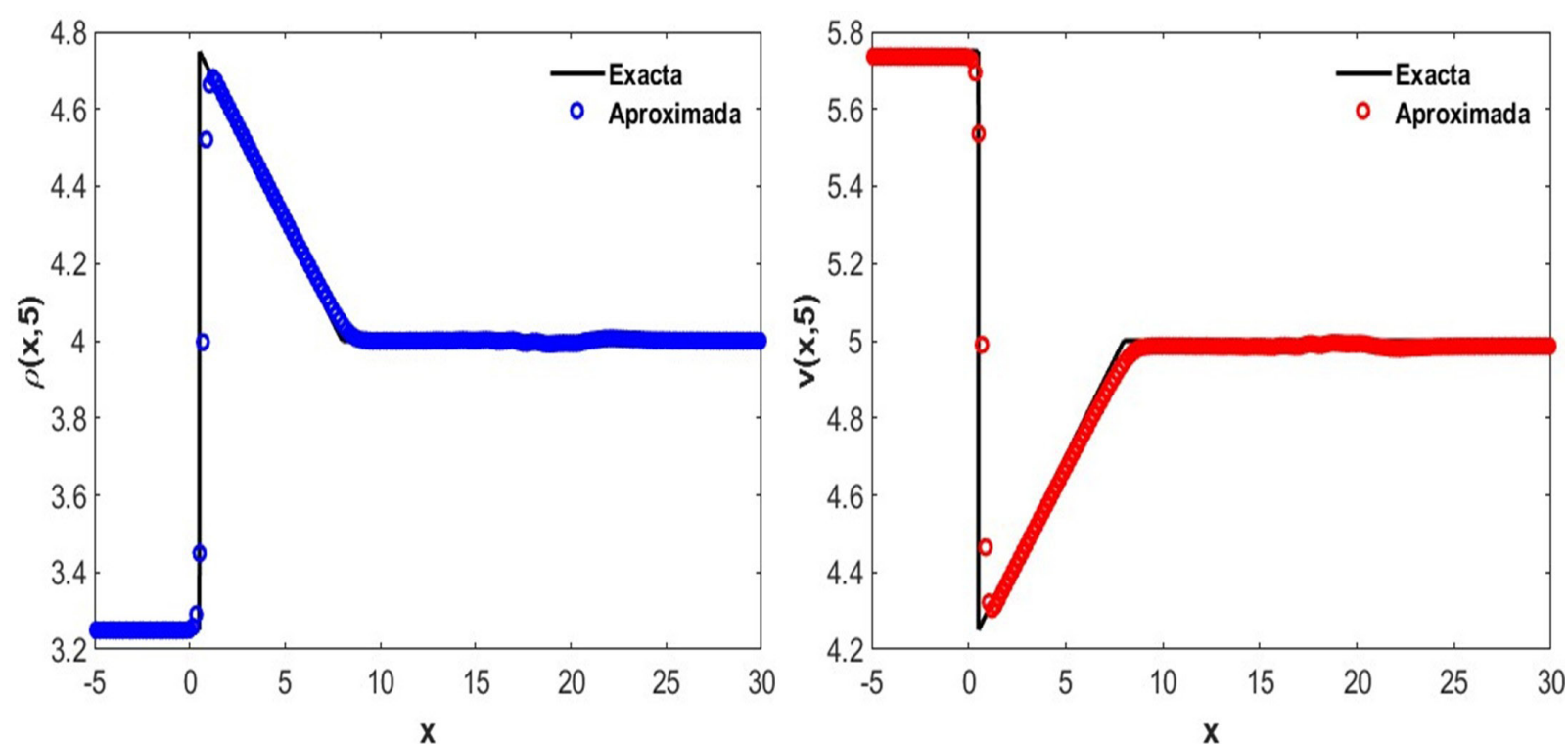


Fig. 4. Comparison between exact and approximate solutions for double Riemann problem at $t = 5$ s.
Source: author.

Next, Table 2 shows the relative errors $E_1(M)$ and $E_2(M)$ in L^1 -norm for different values of M (number of points of spatial mesh). These errors are calculated according to the expressions given in (18):

$$E_1(M) = \frac{\sum j_1^M |\rho_j(t) - \rho(x_j, t)|}{\sum j_1^M |\rho(x_j, t)|}, E_2(M) = \frac{\sum j_1^M |v_j(t) - v(x_j, t)|}{\sum j_1^M |v(x_j, t)|} \quad (18)$$

where $\rho_j(t)$ is the numerical approximation of the exact solution $\rho(x_j, t)$ at the discrete point (x_j, t) given by (17). Note that although the solution has discontinuities, the error decreases consistently as the mesh is refined.

TABLE 2. RELATIVE ERRORS FOR DENSITY AND VELOCITY AT FINAL TIME $T = 5$ S.

M	$E_1(M)$	$E_2(M)$
100	6.128×10^{-3}	7.340×10^{-3}
200	4.978×10^{-3}	5.119×10^{-3}
400	1.859×10^{-3}	1.866×10^{-3}
800	8.751×10^{-4}	8.628×10^{-4}
1600	4.669×10^{-4}	4.287×10^{-4}

Source: author.

Fig 5 displays the errors versus mesh width are shown in logarithmic scale in order to observe the behavior of the convergence rate (estimation of the slope of the straight lines) which is approximately equal to 1.

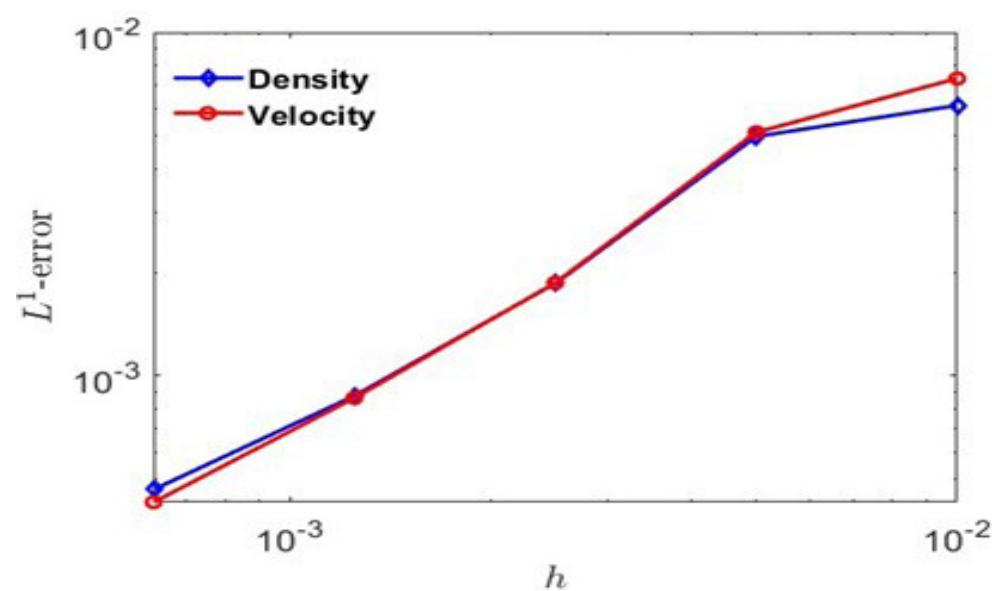


Fig. 5. Approximate errors for density and velocity in logarithmic scale.
Source: author.

CONCLUSIONS

In this paper numerical approximations for the solutions of a second-order traffic flow model were computed using the well-known fifth-order WENO scheme for spatial discretization and a sign-preserving Runge-Kutta exponential method for the temporal discretization obtaining a explicit scheme that does not require taking restrictive time steps for stability. In numerical experiments it was observed that different kinds of waves are correctly captured by the method and the errors exhibit a consistent behavior.

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The Author received the M.Sc. degree in Mathematics from Universidad Nacional de Colombia (Sede Medellín) and Universidad del Norte and the Ph.D. degree in Mathematical Engineering from Universidad de Concepción, Chile, in 2004 and 2010, respectively. Since 2011, I have been a full profesor at the Universidad del Norte in Barranquilla – Colombia.